Extensions of Continuous Selections

Antonio Cruz

Centro de Ciencias Matemticas, UNAM

29 de enero de 2020

Continuous Selections

Definition

Let (X, τ) *be a topological space, and*

$$\mathfrak{F} \subseteq F(X) = \{A \subseteq X \mid X \backslash A \in \tau\} \backslash \{\emptyset\}.$$

$f : \mathfrak{F} \longrightarrow X$ *is called a continuous selection if it is a choice function which is continuous with respect to the Vietoris Topology.*

Continuous Selections

Definition

Let (X, τ) *be a topological space, and*

$$\mathfrak{F} \subseteq F(X) = \{A \subseteq X \mid X \backslash A \in \tau\} \backslash \{\emptyset\}.$$

$f : \mathfrak{F} \longrightarrow X$ *is called a continuous selection if it is a choice function which is continuous with respect to the Vietoris Topology.*

Notation

- Selections who's domain is $[X]^n$ or $[X]^{\leq n}$ will be called n-continuous selections or $_{\leq}n$-continuous selections respectively.
- We write $a \rightarrow_f x$ when $x \in a \in dom(f)$ and $f(a) = x$.

What you need to know

Fix (X, τ)...

More notation

Let f be a continuous selection such that $[X]^n \subseteq dom(f)$ and $\mathcal{U} = \{U_1, \ldots, U_n\} \subseteq \tau$. We write $\mathcal{U} \rightrightarrows U_i$ if and only if whenever we take $y_1 \in U_1, \ldots, y_n \in U_n$, it happens that $\{y_1, \ldots, y_n\} \to_f y_i$

What you need to know

Fix (X, τ)...

More notation

Let f be a continuous selection such that $[X]^n \subseteq dom(f)$ and $\mathcal{U} = \{U_1, \ldots, U_n\} \subseteq \tau$. We write $\mathcal{U} \rightrightarrows U_i$ if and only if whenever we take $y_1 \in U_1, \ldots, y_n \in U_n$, it happens that $\{y_1, \ldots, y_n\} \to_f y_i$

Theorem (folklore)

Let $n \in \omega$ and f be a choice function over $[X]^n$. f is a n-continuous selection if and only if for every $\vec{x} = \{x_1, \ldots, x_n\} \in [X]^n$, there exist $\mathcal{U} = \{U_{x_1}, \ldots, U_{x_n}\} \subseteq \tau$ such that:

- *For every $i \leq n$, $x_i \in U_{x_i}$.*
- *If $i \neq j$ then $U_i \cap U_j = \emptyset$.*
- *$\mathcal{U} \rightrightarrows_f U_i$ if and only if $\vec{x} \to_f x_i$.*

Another Theorem

Theorem (V. Gutev)

For every $n \in \omega\backslash\emptyset$, a space (X, τ) admits a ${}_{\leq}n+1$-continuous selection if and only if it admits a ${}_{\leq}n$-continuous selection and a $n+1$-continuous selection.

Another Theorem

Theorem (V. Gutev)

For every $n \in \omega \setminus \emptyset$, a space (X, τ) admits a $_{\leq}n + 1$-continuous selection if and only if it admits a $_{\leq}n$-continuous selection and a $n + 1$-continuous selection.

Question(M.Hrusak, I. Martnez Ruiz, V. Gutev)

Does there exists a space which admits a 2-continuous selection but not admits a n-continuous selection for some $2 < n \in \omega$.

Another Theorem

Theorem (V. Gutev)

For every $n \in \omega \backslash \emptyset$, *a space* (X, τ) *admits a* $_{\leq}n+1$*-continuous selection if and only if it admits a* $_{\leq}n$*-continuous selection and a* $n+1$*-continuous selection.*

Question(M.Hrusak, I. Martnez Ruiz, V. Gutev)

Does there exists a space which admits a 2-continuous selection but not admits a n-continuous selection for some $2 < n \in \omega$.

Definition (isomorphisms of selections)

Let $n \in \omega$, *and* f, g *be* n*-continuous selections over spaces* X *and* Y *respectively. We say* $\theta : X \longrightarrow Y$ *is an isomorphism from* f *to* g *if and only if it is bijective for every* $a \in [X]^n$, *it happens that* $\theta[f(a)] = g(\theta[a])$

Definition (Regular selections)

A continuous selection f over a space (X, τ) it's called regular if and only if for every $x, y \in X$, it happens that $|f^{-1}[\{x\}]| = |f^{-1}[\{x\}]|$

Definition (Regular selections)

A continuous selection f over a space (X, τ) it's called regular if and only if for every $x, y \in X$, it happens that $|f^{-1}[\{x\}]| = |f^{-1}[\{x\}]|$

Lemma

Let $2 \leq n \in \omega$, and f be a n- continuous selection over X. If k is such that $n \leq k$ and $x \in [X]^k$, then

$$P_n^k(x) := \{y \in [X]^k \mid f|_{[y]^n} \text{ is isomorphic to } f|_{[x]^n}\}$$

is clopen in $[X]^k$.

Lemma (key)

Let $2 \leq n \in \omega$, and f be a $\leq n$- continuous selection over X. If k is such that $n \leq k \leq 2n$ and $x \in [X]^k$ is such that $f|_{[x]^n}$ is not regular, then there exists a continuous selection who's domain is $P_n^k(x)$.

Lemma (key)

Let $2 \leq n \in \omega$, and f be a $\leq n$- continuous selection over X. If k is such that $n \leq k \leq 2n$ and $x \in [X]^k$ is such that $f|_{[x]^n}$ is not regular, then there exists a continuous selection who's domain is $P_n^k(x)$.

Proposition

Let $2 \leq n \in \omega$, and f be a $\leq n$- continuous selection over X. If k is such that $n \leq k \leq 2n$ and for every $x \in [X]^k$ $f|_{[x]^n}$ is not regular, then X admits a k-continuous selection.

Lemma (key)

Let $2 \leq n \in \omega$, and f be a $\leq n$- continuous selection over X. If k is such that $n \leq k \leq 2n$ and $x \in [X]^k$ is such that $f|_{[x]^n}$ is not regular, then there exists a continuous selection who's domain is $P_n^k(x)$.

Proposition

Let $2 \leq n \in \omega$, and f be a $\leq n$- continuous selection over X. If k is such that $n \leq k \leq 2n$ and for every $x \in [X]^k$ $f|_{[x]^n}$ is not regular, then X admits a k-continuous selection.

Lemma

If p is a prime number and and f is a $\leq p$- continuous selection over X, then for every $p \leq k$ such that p does not divides k and for every $x \in [X]^k$ $f|_{[x]^p}$ is not regular.

Theorem

A.C If $n \in \omega$ and X admits a $\leq n$-continuous weak selection , then for every non prime number k between n and $2n$, X admits a k-continuous selection.

Theorem

A.C If $n \in \omega$ and X admits a $\leq n$-continuous weak selection , then for every non prime number k between n and $2n$, X admits a k-continuous selection.

Corollary

If $n \in \omega$, $n+1$ is non prime and X admits a $_{\leq}n$-continuous selection, then X admits a $_{\leq}n+1$-continuous selection

Theorem

A.C If $n \in \omega$ and X admits a $\leq n$-continuous weak selection , then for every non prime number k between n and $2n$, X admits a k-continuous selection.

Corollary

If $n \in \omega$, $n+1$ is non prime and X admits a $_{\leq}n$-continuous selection, then X admits a $_{\leq}n+1$-continuous selection

Corollary

X admits a n-continuous selection for every $n \in \omega$ if and only if it admits a p-continuous selection for every prime number.

The center of the problem

The real remaining question is: What can you say about $P_n^k(x)$ when $f|_{[x]^n}$ is regular?. In fact, here is where most of topological part of the problem really starts and the word selection looses importance.

Thanks for you attention